Estimation of Velocities and Roll-Up in Aircraft Vortex Wakes

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A nonlinear model is developed which determines the swirling and axial velocities in an aircraft vortex wake, given wing lift and drag distributions. The model is shown to reduce to that given by Betz when the axial velocity is the freestream value. The nonlinear interaction of swirling and axial velocities may lead to velocity distributions which are different from those previously calculated. Qualitatively, drag reduces the axial velocity in the vortex and results in an enlarged vortex radius and, therefore, a reduction in swirl velocity. The inviscid model that predicts that significant changes in the structure of the vortex wake, brought about solely by modification of the drag distribution, may require prohibitively large drag penalties. Theoretical results compare favorably with measurements made by Orloff and Grant. A model is developed to estimate the time to roll up a two-dimensional vortex sheet. Results are presented for the cases of linear, parabolic, and elliptic wing loading.

Nomenclature

a = constant, Eq. (21)
A = wing aspect ratio
b = constant, Eq. (21)
$B(t)$ = see Eq. (38)
c = wing chord
c_d = sectional drag coefficient
c_ℓ = sectional lift coefficient
C_L = wing lift coefficient
$g(t)$ = see Eq. (32)
k = constant, Eq. (17)
$\ell(y)$ = sectional lift exerted on the fluid
L = wing lift
P = pressure
q = dynamic pressure
r = radial coordinate
r_t = vortex radius, $r_t = \bar{y}(0)$
$r(t)$ = radial dimension in which the rolled-up vorticity is found
R = characteristic radius of curvature
s = wing semi-span
S = wing planform area
t = time
u,v,w = velocity components in the x,y,z directions, respectively
U_∞ = freestream speed
V = swirl velocity
x,y,z = Cartesian coordinates
$\bar{y}(y)$ = centroid of shed vorticity
$\bar{y}_v(t)$ = horizontal location of the tip vortex during roll-up
α = angle of attack
γ = vortex sheet strength
δ = Eq. (35)
$\Gamma(y)$ = spanwise circulation distribution
Γ' = vortex circulation
Γ_o = wing root circulation
ζ = dummy variable
η = dummy variable
ρ = fluid density
ω = axial component of vorticity

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Index categories: Aircraft Aerodynamics, Jets, Wakes, and Viscid-Inviscid Flow Interactions.

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I. Introduction

Damage to aircraft resulting from encounters with other aircraft wakes is a problem which has been documented in both civil and military aviation. Loss of life and equipment has spurred substantial efforts, in recent years, to determine the hazard associated with vortex wakes. It has become apparent that a more detailed description of the aircraft vortex wake than has hitherto been available is needed. The work reported here is an attempt at this description.

To date, many techniques have been tried in hope of alleviating the wake hazard. Included among these were wing load tailoring [1,2] and various drag devices. [3-5] These techniques have had success in that they appear to deintensify the wake. However, while the relationship between wing load distribution and swirling velocity field is understood, [6-10] the role of drag on wake structure has only been treated to resolve the question of the direction of axial velocity along the vortex centerline. [11-13] When large drag devices, such as spoilers, are used (which may also modify wing loading), the effect on wake structure is not apparent. It is the purpose of this investigation to develop simple models from which quantitative results regarding the roll-up of the vortex wake and the downstream wake structure may be obtained. It will be shown that the drag distribution can significantly alter the structure of the vortex irrespective of viscous processes, but that under cruise conditions this effect is modest. While the models developed here consider only the roll-up of tip vortices, the extension to include "interior" or flap vortices is straightforward and is presented in Ref. 14.

Currently, the techniques available to estimate wake structure may be classified such that they fall into one of three categories‡: 1) the induced drag-kinetic energy of swirl method suggested by Prandtl [16] in 1927; 2) calculation of the motion of discrete vortex elements or filaments first done by Westwater [17-20] in 1935; or 3) roll-up as prescribed by Betz [6] in 1932 involving conservation of an approximate invariant of the fluid motion.

The first technique involves calculation of the swirl kinetic energy per unit length of wake and equating this to the induced drag of the aircraft. The calculation requires an assumption as to the nature of the swirling velocity distribution with sufficient free parameters such that circulation about each core and the impulse of the vortex system is preserved. In Prandtl's calculation, vortices were assumed to have uniform vorticity cores and for an elliptically loaded wing, core radius was obtained to be 0.155 the semispan of the wing. While calculations of this nature are straightforward, they do not

‡The similarity solution found by Kaden [15] is omitted here, since it remains valid only for small times.

give a unique relationship between wing load distribution and vortex velocity fields.

Modeling the vortex sheet shed from the trailing edge of a wing with discrete vortex filaments has been the subject of several investigations. One difficulty is that the similarity solution of Kaden shows that the center of the rolled-up spiral contains an infinite number of turns and, as Westwater points out, can never be modeled by a finite number of vortices. Discrete vortex calculations have given considerable insight into the roll-up problem; however, interpretation of the results obtained to date is still a matter of contention.

The roll-up procedure described by Betz, while available for some time, received little attention until Donaldson[21] showed that the swirl velocity distributions calculated in this way compared most favorably with measurements. Recently, several studies in the spirit of Betz have been undertaken and Donaldson et al.[7] have shown how the roll-up of flap vortices may be calculated according to the Betz assumptions. Experiments have recently shown that the Betz procedure gives a good description of the swirling velocities in the wake, and it is appropriate here to review the Betz model and see if any additional information can be gleaned about the nature of the roll-up phenomenon. It will be shown in Sec. II that the Betz assumptions are readily extended to include axial velocity in the wake, thereby coupling wake structure to both wing lift and drag distributions. In Sec. III, calculations of swirl velocity from the extended Betz model are compared with experiments. A model is developed in Sec. IV, in the spirit of Betz, to estimate the time to roll up a two-dimensional sheet and a comparison is made with calculations performed by Moore.[20]

II. Generalization of the Betz Method to Include Axial Velocity

The method described by Betz for calculating roll-up relates the circulation Γ at wing station y to the circulation Γ' calculated at radius r in an axisymmetric line vortex. The method is based upon the assumption that global invariants of an unbounded, two-dimensional, incompressible, inviscid fluid motion may be applied locally to obtain an approximate description of the vortex structure. The fundamental assumption is that the relationship between the inertia moment of one-half of the vorticity distribution prior to and after roll-up is complete, is approximated by

$$-\int_y^s \frac{d\Gamma(\eta)}{d\eta}[\eta-\bar{y}(y)]^2 dy = \int_o^r \zeta^2 \frac{d\Gamma'(\zeta)}{d\zeta} d\zeta \tag{1}$$

$\bar{y}(y)$ is defined by

$$\bar{y}(y) \equiv -\frac{1}{\Gamma(y)}\int_o^s \frac{d\Gamma(\eta)}{d\eta}\eta d\eta \tag{2}$$

and is the centroid of vorticity shed between stations y and s. Equation (1) is approximate and, as will be shown, can be manipulated so as to allow physical interpretation. With Eq. (1) and a statement of Kelvin's theorem

$$\Gamma(y) = -\int_y^s \frac{d\Gamma(\eta)}{d\eta} d\eta = \int_o^r \frac{d\Gamma'}{d\zeta}(\zeta) d\zeta = \Gamma'(r) \tag{3}$$

which is exact for an inviscid flow, Betz was able to give a rather complicated expression for the swirling velocity in the rolled-up wake of an elliptically loaded wing. By manipulating Eqs. (1-3), Donaldson et al.,[7] Rossow,[9] and Jordan[10] have independently shown the surprisingly simple result: the relationship between r and y is

$$r = \bar{y}(y) - y \tag{4}$$

This result, taken with Eq. (3), states that the value of the circulation at wing station y is the value of the circulation at

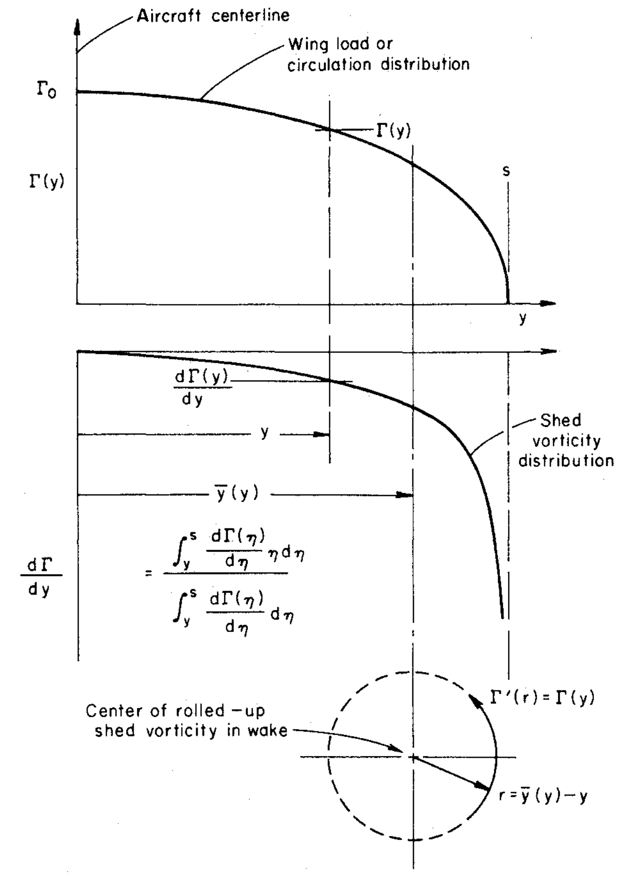

Fig. 1 Illustration of the Betz roll-up model for a simply loaded wing.[7]

radial distance r in an axisymmetric vortex. The radial distance r is equal to the distance from y to the centroid $\bar{y}$ of all the shed vorticity outboard of y. When all the vorticity can be considered rolled-up, the vortex center is located at $y=\bar{y}(0)$ in order to preserve the impulse of the flow. Since, at this point, r also equals $\bar{y}(0)$, the circular regions containing vorticity just touch along the aircraft centerline. Figure 1 depicts the roll-up relationships.

Physical insight into the roll-up equations (1-4), which suggest an extension of the Betz model to include axial velocity, is obtained by performing the following manipulations. Equation (1) is multiplied by $-\rho U_\infty/2$ and integrated by parts to yield

$$-\rho\frac{U_\infty}{2}\{\Gamma(y)[y-\bar{y}(y)]^2-\Gamma'(r)r^2\} + \int_y^s \ell(\eta)[\eta-\bar{y}(y)]d\eta = 2\pi\rho U_\infty\int_o^r V(\zeta)\zeta^2 d\zeta \tag{5}$$

where $\ell(y) = -\rho U_\infty \Gamma(y)$ is the sectional wing loading exerted on the fluid and V the swirl velocity in the vortex. The first term in Eq. (5) vanishes when Eqs. (3) and (4) are substituted. The remaining terms prescribe the distribution of angular momentum in the vortex. The Betz model therefore distributes the angular momentum such that the torque exerted by the wing [calculated about $\bar{y}(y)$] between y and s equals the flux of angular momentum through a circle of radius r.

In a straightforward manner, nonuniform velocity in the vortex may be included in the Betz model by modifying Eq. (5) to read

$$\int_y^s \ell(\eta)[\eta-\bar{y}(y)]d\eta = 2\pi\rho\int_0^r u(\zeta)V(\zeta)\zeta^2 d\zeta \tag{6}$$

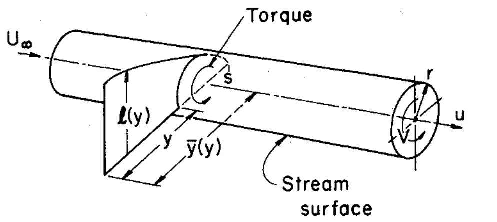

Fig. 2 Geometry of the roll-up model to include axial velocity.

The geometry of the flow model might be thought to be that illustrated in Fig. 2. By differentiating Eq. (6), and using Eqs. (2) and (3), it can be shown that

$$\frac{u(r)}{U_\infty}\frac{\mathrm{d}r^2}{\mathrm{d}y}=\frac{\mathrm{d}}{\mathrm{d}y}[\bar{y}(y)-y]^2 \tag{7}$$

The result given in Eq. (4) is obtained with $u(r)=U_\infty$.

Before coupling the wing drag distribution to the axial velocity in the vortex, it is possible to show how the axial velocity modifies the velocity $V(0)$ at the center of the vortex. Assuming that $u(0)$ is finite, Eq. (7) may be integrated for small r to yield

$$r=\left[\frac{U_\infty}{u(0)}\right]^{1/2}[\bar{y}(y)-y] \tag{8}$$

as $r\rightarrow 0$. The swirling velocity is $V(r)=\Gamma'(r)/2\pi r$ or when $r\rightarrow 0$

$$V(0)=\frac{1}{2\pi}\left[\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}r}\right]_s \tag{9}$$

where it has been assumed that $(\mathrm{d}\Gamma/\mathrm{d}y)_s$ is bounded. Differentiating Eq. (8) and taking the limit using Eq. (2), yields

$$V(0)=\frac{-1}{\pi}\left[\frac{u(0)}{U_\infty}\right]^{1/2}\left[\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right]_s \tag{10}$$

Deficits in axial velocity $[u(0)/U_\infty<1]$ therefore result in a reduction of the centerline swirl velocity.

We may now turn our attention to coupling the axial velocity in the vortex to the wing drag distribution. Making an axial momentum balance across a cylindrical control volume of radius r, containing the portion of the wing outboard of station y, yields

$$\int_y^s c_d(\eta)c(\eta)q\,\mathrm{d}\eta + 2\pi\int_0^r[P+\rho u(u-U_\infty)]\zeta\mathrm{d}\zeta=0 \tag{11}$$

where the axial velocity of the fluid fluxing through the cylindrical surface is approximated by U_∞. The pressure far upstream has been taken to be zero and $c_d(y)c(y)q$ is the wing sectional drag. When the u^2 term is linearized, Eq. (11) is that given by Brown.[11] Equation (11) is written in the same spirit as Eq. (1), since it assumes that the wing drag distributes itself in the rolled-up vortex in the same manner as the shed axial vorticity. As discussed by Brown, the assumption is a natural one since the axial vortex lines and the viscous wake are one and the same. However, it is clear that the contribution of the pressure on the area external to the regions containing axial vorticity is neglected. Therefore, while Eq. (11) correctly accounts for the profile portion of drag on the wing, the effect of induced drag is approximated. Differentiating Eq. (11) and substituting Eq. (7) yields

$$c_d(y)c(y)=(\pi/q)[P+u(u-U_\infty)](\mathrm{d}/\mathrm{d}y)(\bar{y}-y)^2 \tag{12}$$

The noniform pressure in the vortex is primarily a result of the swirl and may, therefore, be calculated from

$$P=-\frac{\rho}{4\pi^2}\int_r^\infty\frac{\Gamma'^2}{\zeta^3}\mathrm{d}\zeta \tag{13}$$

Equations (7, 12, and 13) are a nonlinear system with the boundary conditions

$$\left.\begin{array}{ll} P=-\rho/2\,V^2 & \text{at } y=0 \\ r=0 & \text{at } y=s \end{array}\right\} \tag{14}$$

Together, Eqs. (7) and (12-14) determine the axial and swirl velocities in a wake vortex given the half-wing lift and drag distributions. Nonlinearity and the nature of the boundary conditions dictate that, in general, solutions will have to be obtained numerically.

Before proceeding to obtain solutions, it is possible to obtain the behavior of the velocities for small r. It may be shown that for an elliptically loaded wing as $r\mapsto 0$

$$[u(r)/U_\infty]=(2/3)k^{2/3}(r/s)^{-2/3} \tag{15}$$

$$[V(r)2\pi s/\Gamma_0]=3^{1/2}k^{1/6}(r/s)^{-2/3} \tag{16}$$

where

$$k=\left[\frac{C_L^2}{\pi^3A^2}-\frac{(c_dc)_s}{36s}\right]\frac{81}{4\pi} \tag{17}$$

In obtaining Eqs. (15) and (16), it is assumed $(c_dc)_s$ is finite. When the axial velocity is not coupled to the swirl, Brown has shown that for small r

$$\frac{V(r)2\pi s}{\Gamma_o}=(2r/3s)^{-1/2} \tag{18}$$

and that the axial velocity is proportional to the reciprocal of r. However, the constant of proportionality is incorrect due to neglecting the head loss in the vortex that is associated with distributing the axial vorticity as specified by Betz. When proper account is taken of head loss, the result is

$$\frac{u(r)}{U_\infty}=\frac{3}{8\pi}\left[\frac{\Gamma_o^2}{\pi U_\infty^2s^2}-\frac{(c_dc)_s}{s}\right]\frac{s}{r} \tag{19}$$

and may be compared with Brown's Eq. (23). The results of the local solution necessitate two comments regarding the need to include viscosity in a region about the vortex center and the tightness of the rolled-up vortex.

Moore and Saffman[13] have recently shown that the singularities in velocities result as a consequence of the inviscid assumption and may be removed by a viscous correction. We, therefore, do not concern ourselves with the details of this correction and only comment on the significance of this region. As argued by Moore and Saffman, the radial dimension over which viscous effects cannot be neglected is very small [being of order $(\nu x/U_\infty)^{1/2}$ compared to the dimension characterizing the complete vortex. The amount of angular momentum in this region commonly called the viscous core or simply core is also small compared to the total angular momentum in the vortex. Therefore, the error introduced by neglecting viscous corrections when calculating quantities which require integration over the whole of the vortex (i.e., rolling moments) will be small. This, of course, is not to say that the core of the vortex can be neglected in general; for instance, Donaldson and Bilanin[22] have shown that the core plays an important negative role in the turbulent dissipation of a vortex.

Referring to Eqs. (16) and (18), it is seen that the coupled model predicts a more concentrated vortex than the linear model (based on the strength of the singularity in the swirl velocity at the vortex center). While in itself this result is not particularly significant, it should remind us that the hazard associated with a particular wake is associated with the torque which may be induced on an encountering aircraft. This torque is related to the flux of angular momentum in the wake. Under the assumptions used to develop the model presented here, the flux of angular momentum in a Betz vortex is only a function of the lift distribution. Wing drag does not change the magnitude of this flux, only its distribution.

III. Calculation of Vortex Structure and Comparison with Experimental Measurement

It is appropriate to first find simple analytic solutions which illustrate the relationship between wing lift and drag distributions and downstream vortex structure before making a comparison with experiments. Unfortunately, the direct problem of specifying the lift and drag distributions does not appear to yield analytic solutions. However, numerical integration of the direct problem is now routine and examples may be found in Refs. 22 and 23. The indirect problem, specifying the axial velocity and lift distribution and determining the drag distribution and swirling velocity, is straightforward for simple distributions.

Equation (7) can be integrated if the wing loading is linear and, therefore, of the form

$$\Gamma = \Gamma_o [1-(y/s)] \quad (20)$$

and the axial velocity is given by

$$u = U_\infty [a + b(r/r_t)^2] \quad (21)$$

The constants a and b may be chosen such that u is positive; therefore, $a \geq 0$. Negative axial velocities imply a flux of angular momentum from downstream and, therefore, violate the assumptions implicit in Eq. (6). The radius of the region containing all the vorticity shed between the wing root and tip r_t is to be determined. Integrating Eq. (7) yields

$$(s-y)^2 = 4ar^2 + (2br^4/r_t^2) \quad (22)$$

When $r=r_t$, $y=0$ and therefore

$$r_t/s = [4a+2b]^{-1/2} \quad (23)$$

The Betz result is obtained with $a=1$, $b=0$; all the vorticity is contained within a circle having radius $s/2$. Referring to Eq. (21), sufficient conditions for an axial velocity excess in the wake occur when $a>1$ and $b \geq 0$; r_t decreases, and the vortex is intensified. Axial velocity defects are associated with increases in r_t and, therefore, more diffuse vortices.

The circulation and swirl velocity distributions in the vortex are given by

$$\Gamma'/\Gamma_o = \{a + [b(r/r_t)^2/2]\}^{1/2} 2r/s \quad (24)$$

$$V2\pi s/\Gamma_o = \Gamma' s/\Gamma_o r \quad (25)$$

The drag on the wing, shown in Fig. 3, is calculated from Eq. (12) for 5 cases. At an average drag coefficient of about 0.012 (a typical value), the axial velocity in the vortex is uniform and equal to the freestream value (Case 3). The vortex radius is $0.5s$ and is taken to define a reference circular area. By increasing the average drag coefficient by nearly an order of magnitude to 0.11, the vortex radius is increased 16% to $0.585s$ (Case 2). The flux of angular momentum through the reference area, when compared with Case 3, is a measure of the deintensification which can be achieved by increased drag. The calculation shows that the flux in Case 2 is reduced a highly desirable 43%. However, the model suggests that vortex deintensification brought about solely by this technique will have prohibitively large drag penalties. Determination of the actual deintensification which may be brought about by increased drag will, however, have to include estimates of the increased turbulent diffusion which will also result. The problem of the least intense vortex for a given lift and drag coefficient is surely worthy of additional study. Cases 4 and 5 show the effect of distributed thrust on vortex structure. Curiously, the effect of thrust is to intensify the vortex; a result which is a consequence of the inviscid nature of the model.

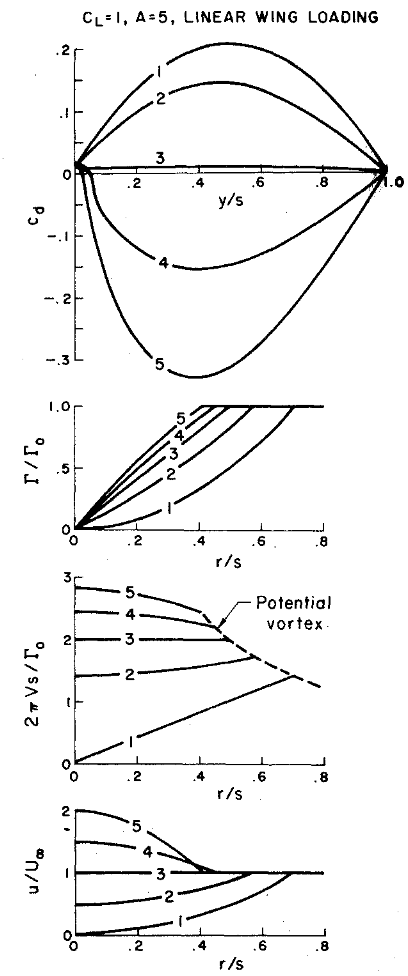

Fig. 3 The vortex wake structure for a linearly loaded wing.

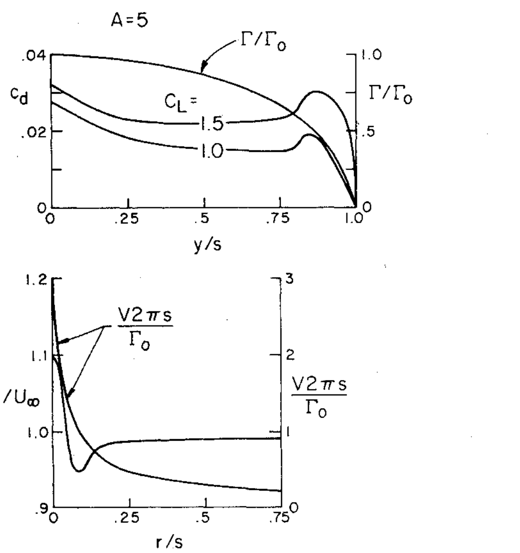

Fig. 4 Examples of drag distributions which result in axial velocity profiles which have both an excess and defect.

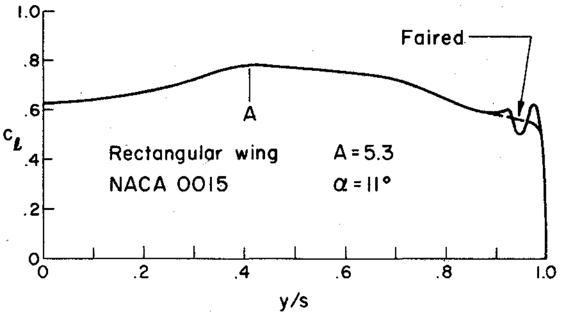

Fig. 5 Scaled sectional lift coefficient for a half-wing model (from Chigier and Corsiglia).

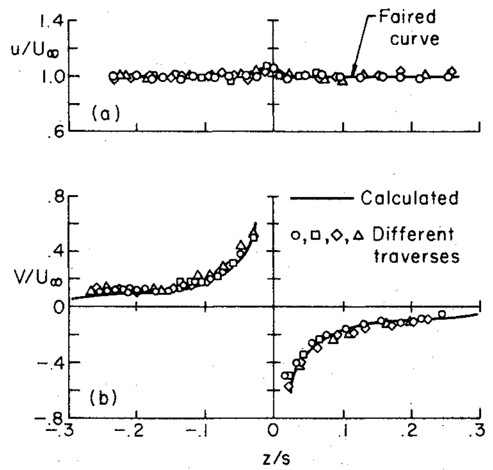

Fig. 6a) Measured axial velocity distribution and b) measured and predicted swirl velocity distribution (clean wing).

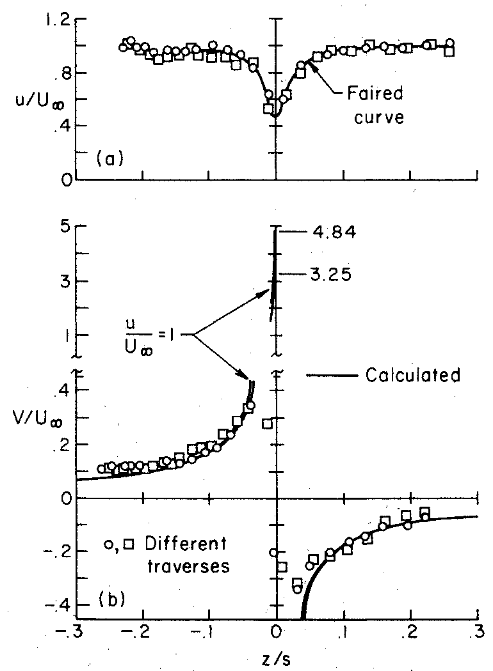

Fig. 7a) Measured axial velocity distribution and b) measured and predicted swirl velocity distribution (small spoiler at wing tip).

Axial velocities in the vortex may be either an excess or defect depending on the wing lift and drag distributions. In addition, it is quite possible that axial velocity distributions can result which have excesses over certain radial intervals and defects over others. Figure 4 illustrates such a situation. Tailoring the drag distribution so that strong radial gradients in axial velocity develop may prove to be an effective means of enhancing turbulent decay.

At this time, a complete comparison of the model with experiment is not possible since the simultaneous measurement of wing lift and drag distributions and vortex axial and swirling velocities has not, to our knowledge, been made. The model can be checked in part, however, with the measurements by Orloff and Grant.[24] They have used a laser Doppler velocimeter to measure the velocities in the wake of a model half-wing. The same wing was used in an earlier investigation by Chigier and Corsiglia,[25] who presented the lift distribution determined from integration of surface pressure data. The lift and axial velocity distributions can then be used in Eq. (7) to calculate $r(y)$. This result, with Eq. (3), determines the swirl velocity distribution which may then be compared with the experimentally measured values.

In Fig. 5, the sectional lift coefficient scaled to the test angle of attack of 11° is shown. The lift distribution is not simple and is one which will roll up in two vortices: a tip vortex and a fuselage vortex of opposite sign (provided the oscillations near the tip may be neglected). A criterion which determines how the lift distribution will divide itself has been given in Ref. 7. Yates[26] has recently given this criterion firm theoretical support by direct calculation. The vorticity shed outboard of station A rolls up into a tip vortex.

Orloff and Grant obtained data for two wing configurations: the clean wing and the same wing with a small spoiler installed at the tip at the 1/4-chord position. In Fig. 6, the model is compared with data obtained seven chords downstream of the clean wing. The axial velocity is nearly the freestream value and the calculated swirl velocities are essentially those given by the Betz model. The calculation is in good agreement outside the vortex core. In Fig. 7, the calculated swirl velocities are compared with data obtained seven chords behind the wing with the small spoiler installed. For this configuration, the C_L of the wing is reportedly reduced 4%. The spoiler obviously changes the lift distribution, but this change was not measured. The correction is presumably small and is taken into account in the calculation by scaling the sectional lift coefficient. The agreement with the measured swirling velocity is again good outside the core. The effect of the spoiler apparently is to age the core of the vortex more rapidly. The calculation with $u/U_\infty = 1$ is also shown. It is at first glance surprising that with an axial velocity defect of 50% at the vortex center, so small a change in the swirl velocity distribution results. This is understood by reexamining Eq. (7) in the form

$$\rho u(r) 2\pi \mathrm{d}r = \rho U_\infty \left[2\pi \frac{\mathrm{d}(\bar{y}-y)^2}{\mathrm{d}y} \mathrm{d}y \right] \tag{26}$$

The result is a statement regarding the conservation of mass. The right-hand side of Eq. (26) is the mass flux through a rectangle having width $\mathrm{d}y$ and height $-2\pi\mathrm{d}(\bar{y}-y)^2/\mathrm{d}y$ and is only a function of the wing lift distribution. The left-hand side is the mass flux through the annular area $2\pi r\mathrm{d}r$. Therefore, the swirl velocity is expected to change in a circular area when the mass flux through this area is significantly altered. In Fig. 7, the mass deficit in the area of radius $r/s = 0.3$ is 2% less than the mass through this area when the axial velocity is the freestream value. Judging from the sample calculation whose results are presented in Fig. 3, more significant vortex deintensification will require a much larger drag penalty.

IV. Estimated Time to Roll Up a Two-Dimensional Sheet

The assumptions made by Betz can be used to develop a model to estimate the time required to roll up the vorticity behind a simply loaded wing. In the spirit of Betz, we consider the roll-up of a two-dimensional vortex sheet of strength $\gamma(y,t)$. Referring to Fig. 8, we assume that vorticity which has been convected past station A is rolled up in the vortex and is distributed according to Eq. (4). Furthermore, we assume that the portion of the sheet which is not yet in the vortex remains horizontal as shown. The error which is a consequence of this assumption is to be discussed.

The time rate of change of circulation in the vortex is given by

$$(\mathrm{d}\Gamma'/\mathrm{d}t) = [v - (\mathrm{d}\bar{y}_v/\mathrm{d}t)]\gamma(\bar{y}_v, t) \tag{27}$$

where v is the horizontal velocity at station A induced by the vorticity, and γ is the vortex sheet strength at station A. The term $\mathrm{d}\bar{y}_v/\mathrm{d}t$ accounts for the inward motion of the vortex as roll up proceeds.

In the evaluation of the sheet strength, it is necessary to account for sheet stretching which is the result of nonuniform convection along the sheet. To obtain an expression governing the stretching, the two-dimensional, inviscid vorticity equation

$$(\partial\omega/\partial t) + v(\partial\omega/\partial y) + w(\partial\omega/\partial z) = 0 \tag{28}$$

is integrated through the sheet from $z = -\infty$ to $z = \infty$. After integrating by parts and using the continuity equation, we obtain

$$(\partial\gamma/\partial t) + \partial/\partial y) \int_{-\infty}^{\infty} v\omega \mathrm{d}z = 0 \tag{29}$$

where

$$\gamma = \int_{-\infty}^{\infty} \omega \mathrm{d}z$$

The horizontal velocity v is the sum of two terms v_c and v_s; v_c is the velocity due to the rolled-up portion of the sheet and is given by

$$v_c = (z\Gamma'/2\pi)\{-[(y-\bar{y}_v)^2 + z^2]^{-1} + [(y+\bar{y}\bar{y}_v)^2 + z^2]^{-1}\} \tag{30}$$

Since the sheet is of negligible thickness, Eq. (29) may be written

$$(\partial\gamma/\Theta t) + (\partial/\partial y)\ (v_c\gamma) = (-\partial/\partial y) \int_{-\infty}^{\infty} v_s\omega \mathrm{d}z \tag{31}$$

When a vortex sheet is planar, v_s is identically zero which motivated the assumption regarding the geometry of the sheet. The error introduced is not too severe, however, since had the sheet been allowed to deform having a characteristic radius R, the horizontal induced velocities would be of order Γ_s/R where Γ_s is the circulation of the sheet. Initially, R is infinite. When roll-up has proceeded for some time, R is finite; however, Γ_s is small since most of the circulation has already been rolled into the vortex. We therefore neglect the right-hand side of Eq. (31). This approximation implies that the roll-up phenomenon is strongly dominated by the developing vortex. The approximations introduced thus far will underestimate the time to roll-up since sheet stretching is also underestimated.

Unfortunately, even with these simplifications, the solution of Eq. (31) with v_c given by Eq. (30) is quite difficult. Therefore, the functional form of the convecting velocity is chosen so that it is possible to find an analytic solution to Eq. (31) and yet retain the physics of the stretching phenomenon. One such velocity is

$$v_c = g(t)y \tag{32}$$

where

$$g(t) = \Gamma'/2\pi r(t)\bar{y}_v \tag{33}$$

The general solution of Eq. (31) can now be found by the method of characteristics and is

$$\gamma(y,t) = \Gamma_o f(\delta)/y \tag{34}$$

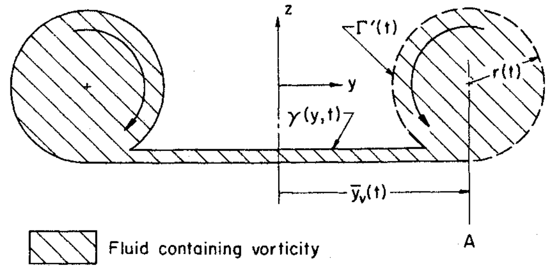

Fig. 8 Simple two-dimensional roll-up model.

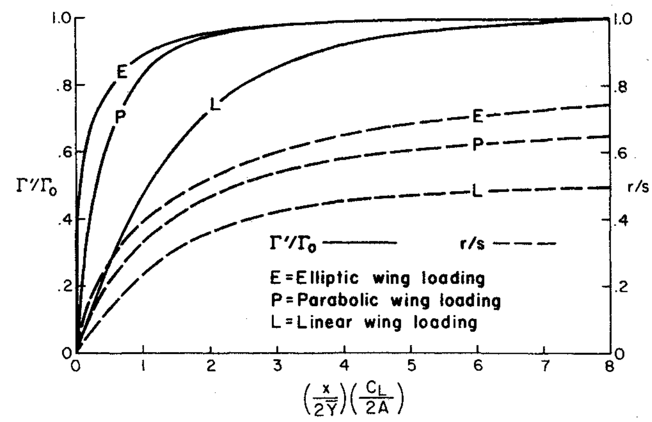

Fig. 9 Vortex circulation and radius as a function of downstream distance (time has been replaced by x/U_∞).

where

$$\delta = (y/s)\ \exp\ (-\int_o^t g(t)\mathrm{d}t) \tag{35}$$

f is an arbitrary function of δ and is determined from the initial sheet strength distribution.

We will present results for the linear, parabolic and elliptical wing loading. The initial sheet strengths are calculated from $\gamma = -\mathrm{d}\Gamma/\mathrm{d}y$ and are

$$\gamma s/\Gamma_o = \begin{cases} 1 & \text{Linear} \\ 2y/s & \text{Parabolic} \\ [1-(y/s)^2]^{-1/2} y/s & \text{Elliptic} \end{cases} \tag{36}$$

The vortex sheet strength at Station A as a function of time is

$$\gamma s/\Gamma_o = \begin{cases} B(t) \\ 2B(t)^2\bar{y}_v/s \\ \{1-[B(t)\bar{y}_v/s]^2\}^{-1/2} B(t)^2\bar{y}_v/s \end{cases} \tag{37}$$

where

$$B(t) = \delta s/y \tag{38}$$

The location of the developing vortex $\bar{y}_v(t)$ is not known and is determined from the conservation of impulse. The calculation appears tractable for only the linear load case where

$$\bar{y}_v(t)/s = [1 + (\Gamma'/\Gamma_o)]^{-1} \tag{39}$$

However, a good approximation to $\bar{y}_v$ is to use $\bar{y}$ as defined in Eq. (2) or rewritten as an explicit function of Γ as

$$\bar{y}(\Gamma) = y(\Gamma) - \frac{1}{\Gamma} \int_o^\Gamma \Gamma \left[\frac{\mathrm{d}\Gamma}{\mathrm{d}y} \right]^{-1} \mathrm{d}\Gamma \tag{40}$$

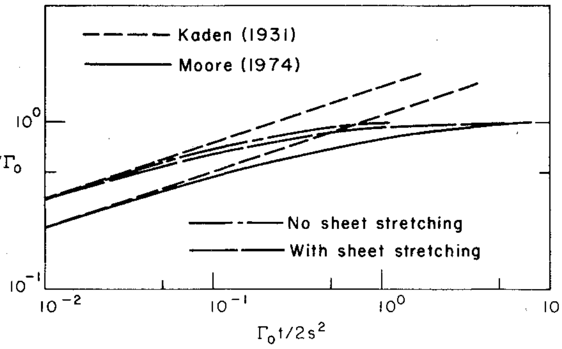

Fig. 10 Comparison of the simple roll-up model with the detailed calculations of Moore. (The constant in Kaden's solution has been adjusted to give agreement at $\Gamma_o t/2s^2 = 10^{-2}$.)

For the linear load

$$\bar{y}/s = 1 - (\Gamma/2\Gamma_o) \tag{41}$$

When Eq. (41) is used to approximate $\bar{y}_v$, an error of 6.25% in the impulse for the linear load occurs during the roll-up. The error is less than this value for life distributions more highly loaded at the tip and is therefore not significant.

For wings having a linear load distribution, an ordinary nonlinear integrodifferential equation describing the time rate of growth of the vortex is

$$\frac{d\Gamma'}{dt} = -\frac{1}{\pi}\left[\frac{\Gamma_o}{s}\right]^2$$

$$\times \left\{\frac{1}{2} - \exp\left[\int_o^t \frac{2}{\pi}\,\frac{\Gamma_o}{s^2}\left[2 - \frac{\Gamma'}{\Gamma_o}\right]^{-1} dt\right\}^{-1} \tag{42}$$

The solution is

$$\Gamma'/\Gamma_0 = 1 - \exp\left(-2\Gamma_0 t/\pi s^2\right) \tag{43}$$

The equations governing the roll-up for the parabolic elliptically loaded wing are considerably more complicated and are not presented here. The solutions for these cases must be determined numerically and are shown along with the linear case in Fig. 9.

Spreiter and Sacks[27] have also made estimates of the downstream distance at which the vortex can be considered to be essentially rolled-up by applying the results obtained by Kaden for a semi-infinite wing. For elliptic wing loading, they obtained the nondimensional downstream distance of $xC_L/4\bar{y}A = 0.18$ which corresponds to having approximately 55% of the wing root circulation in the vortex (as calculated here). The discrepancy is believed to arise as a consequence of using Kaden's solution where it is not strictly valid.

For small times the solutions can be shown to behave as

$$\Gamma'/\Gamma_o = \begin{cases} 2t\Gamma_o/\pi s^2 & \text{Linear} \\ 8t\Gamma_o/\pi s^2 & \text{Parabolic} \\ 3(t\Gamma_o/4\pi s^2)^{1/3} & \text{Elliptic} \end{cases} \tag{44}$$

The $t^{1/3}$ behavior for the elliptic load agrees with the similarity solution obtained by Kaden.

In Fig. 10, our results are compared with those recently obtained by Moore.[20] Part of the discrepancy has already been explained by the approximations which have been made. However, an additional point is that Moore has chosen the station past which the sheet is to be considered rolled-up at a location 90° in the counterclockwise sense from station A in Fig. 8. His results are, therefore, biased to be lower than those obtained here. The amount is difficult to calculate, but the difference is expected to be most significant for small times. The result neglecting sheet stretching is obtained by taking v_c constant in Eq. (31), equal to the horizontal velocity at station A in Fig. 8. In this case, complete roll-up occurs in finite time.

V. Conclusions

A simple, nonlinear model in the spirit of Betz has been developed to demonstrate the role of wing lift and drag distributions on vortex wake structure. While the axial and swirling velocities in the wake may be strongly coupled, estimates of the drag increase required to significantly alter the wake structure appear prohibitively large. Qualitatively, drag increases the vortex diameter by spreading the flux of angular momentum over a greater radius and, therefore, reduces the intensity of the vortex. Axial velocities may be either towards the wing or away from the wing, depending on the details of the wing lift and drag distributions. Singularities in axial and swirling velocities may occur, and they must be resolved with viscous corrections.

The roll-up time of a two-dimensional vortex sheet is investigated within the framework of the Betz approximations. The results compare favorably with more complicated calculations. The models developed were for simply loaded wings. The extension to wing lift distributions which result in two or more discrete vortices, while not presented here, is straightforward.

References

[1] Bilanin, A. J. and Widnall, S. E., "Aircraft Wake Dissipation by Sinusoidal Instability and Vortex Breakdown," AIAA Paper 73-107, Washington, D.C., 1973.

[2] Uzel, J. N. and Marchman, J. F., "The Effect of Wing-Tip Modifications on Aircraft Wake Turbulence," Rept. VPI-E-72-8, July 1972, Virginia Polytechnic Institute, Blacksburg, Va.

[3] Corsiglia, V. R., Schwing, R. G., and Chigier, N. A., "Rapid Scanning, Three-Dimensional Hot-Wire Anemometer Surveys of Wing-Tip Vortices," *Journal of Aircraft,* Vol. 10, Dec. 1973, pp. 752-757.

[4] Patterson, J. C., "Lift-Induced Wing-Tip Vortex Attentuation," AIAA Paper 74-38, Washington, D.C., 1974; to be published in *Journal of Aircraft,* Vol. 12, Aug. 1975.

[5] Corsiglia, V.R., Jacobsen, R.A., and Chigier, N., "An Experimental Investigation of Trailing Vortices Behind a Wing with a Vortex Dissipator," *Aircraft Wake Turbulence and Its Detection,* Plenum, New York, 1971, pp. 229-242.

[6] Betz, A., "Behavior of Vortex Systems," TM 713, NACA; *Zeitschrift fuer Angewandte Mathematik und Mechank.* transl. from Vol. XII. 3, 1932.

[7] Donaldson, C. duP., Snedeker, R. S., and Sullivan, R. D., "A Method of Calculating Aircraft Wake Velocity Profiles and Comparison with Full-Scale Experimental Measurements," *Journal of Aircraft,* Vol. 11, Sept. 1974, pp. 547-555.

[8] Mason, W. H. and Marchman, J. F., "Farfield Structure of an Aircraft Trailing Vortex," *Journal of Aircraft,* Vol. 10, Feb. 1973, pp. 86-92.

[9] Rossow, V., "On the Inviscid Rolled-Up Structure of Lift-Generated Vortices," *Journal of Aircraft,* Vol. 10, Nov. 1973, pp. 647-650.

[10] Jordan, P. F., "Structure of Betz Vortex Cores," *Journal of Aircraft,* Vol. 10, Nov. 1973, pp. 691-693.

[11] Brown, C. E., "Aerodynamics of Wake Vortices," *AIAA Journal,* Vol. 11, April 1973, pp. 531-536.

[12] Batchelor, G. K., "Axial Flow in Trailing Line Vortices," *Journal of Fluid Mechanics,* Vol. 20, Pt. 4, 1964, pp. 645-658.

[13] Moore, D. W. and Saffman, P. G., "Axial Flow in Laminar Trailing Vortices," *Proccedings of the Royal Society: Ser. A: Mathematical and Physical Sciences* Vol. 333, 1973, pp. 491-508.

[14] Bilanin, A. J., Donaldson, C. duP., and Snedeker, R.S., "An Analytic and Experimental Investigation of the Wakes Behind Flapped and Unflapped Wings," Rept. AFFDL-TR-74-90, July 1974, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio.

[15] Kaden, H., "Aufwicklung einer un-stabilen Unstetigkeitsflache," *Ingenieur-Archiv.,* Bd. II, 1931, pp. 140-168.

[16] Durand, W.F., ed., *Aerodynamic Theory,* Vol. 2, Division E, Durand Reprinting Committee, California, 1943, pp. 328-330.

[17] Westwater, F.L., "Rolling Up of the Surface of Discontinuity Behind an Aerofoil of Finite Span," R&M 1962, 1935, Aeronautical Research Council, London.

[18] Hackett, J. E. and Evans, M. R., "Vortex Wakes Behind High Lift Wings," *Journal of Aircraft,* Vol. 8, May 1971, pp. 334-340.

[19] Landgrebe, A. J. and Cheney, M. C., "Rotor-Wakes—Key to Performance Prediction," presented at the Symposium on Status of

Testing and Modeling Techniques for V/STOL Aircraft, Mideast Region of the American Helicopter Society, Oct. 1972.

[20] Moore, D.W., "A Numerical Study of the Roll-Up of a Finite Vortex Sheet," *Journal of Fluid Mechanics,* Vol. 63, Pt. 2, 1974, pp. 225-235.

[21] Donaldson, C. duP., "A Brief Review of the Aircraft Trailing Vortex Problem," Rept. AFOSR-TR-71-1910, July 1971, Air Force Office of Scientific Research, Arlington, Virginia.

[22] Donaldson, C. DuP. and Bilanin, A. J., "Vortex Wakes of Conventional Aircraft," AGARDograph 204, May 1975.

[23] Bilanin, A. J., Donaldson, C. duP., and Snedeker, R. S., "Calculation of the Inviscid Wake of a Boeing 747 Aircraft," Rept. 224, Aug. 1974, Aeronautical Research Associates of Princeton, Inc., Princeton, N.J.

[24] Orloff, K.L., and Grant, G.R., "Trailing Vortex Wind-Tunnel Diagnostics with a Laser Velocimeter," *Journal of Aircraft,* Vol. 11, Aug. 1974, pp. 447-482.

[25] Chigier, N. A. and Corsiglia, V. R., "Tip Vortices; Velocity Distributions," TM X-62, 087, Sept. 1971, NASA.

[26] Yates, J. E., "Calculation of Initial Vortex Roll-Up in Aircraft Wakes," *Journal of Aircraft,* Vol. 11, July 1974, pp. 397-400.

[27] Spreiter, J. R. and Sacks, A. H., "The Rolling Up of the Trailing Vortex Sheet and Its Effect on the Downwash Behind Wings," *Journal of the Aerospace Sciences,* Vol. 18, Jan. 1951, pp. 21-32.